Characterizing dominant strategy implementation in continuous domains*

Juan Carlos Carbajal
School of Economics
University of Queensland
jc.carbajal@uq.edu.au

Rabee Tourky
School of Economics
University of Queensland
rabee@purnima.net

May 4th, 2009

Abstract

We consider the design of dominant strategy mechanisms in social choice environments with quasi-linear preferences, private valuations, and arbitrary metric spaces as social choice sets. We restrict the type space of the agents to be the space of continuous valuation functions define on the choice set. Within this framework, we provide a complete characterization of dominant strategy mechanisms. That is, we show that any dominant strategy direct mechanism is composed of (i) a social choice rule that is a random dictatorship, in that it maximizes a weighted sum of individual valuations plus an external contribution function; (ii) a transfer scheme that is a generalization of the Groves transfer scheme. We provide a strong connection between this result and the general impossibility result of Gibbard and Satterthwaite, as presented by Barberà and Peleg (1990). Further, we show that imposing any budgetary restriction to net transfers to society reduces the scope of dominant strategy implementation, in that, for an important family of transfers, only dictatorial social choice rules are compatible with any budgetary restriction.

Keywords: dominant strategy implementation; continuous domain; random dictatorship; Groves transfers; Gibbard-Satterthwaite theorem; revenue equivalence.

1 Introduction

The impossibility theorem discovered by Gibbard [5] and Satterthwaite [12] constitutes one of the most remarkable results in social choice theory. Roughly speaking, this result imposes a striking limitation on the way a society makes collective decisions that are not prone to manipulation: with universal preference domains, any strategy-proof social

*Preliminary version; comments welcome.

choice function must be dictatorial. Restricting the domain of individual preferences over social outcomes is of course a way out. One of the most successful restrictions explored in the literature is to assume the existence of a transferable, divisible commodity (i.e., money) that enters the utility function in a linear fashion. With the presence of a transferable commodity and quasi-linear utilities over social alternatives and money, it is possible to design transfer schemes that implement non-dictatorial social choice rules in dominant strategy.

This does not mean that every social choice rule (SCR) is dominant strategy implementable in quasi-linear environments, at least not in non-trivial social choice frameworks. Thus, after an environment has been specified in terms of the set of agents in the economy, the set of social alternatives, the domain of individual preferences and the informational structure, one can ask the question of what are the properties of a SCR that characterize implementation; more succinctly, what types of choice rules are dominant strategy implementable. On the other hand, if a given SCR is known to be dominant strategy implementable in a certain environment, then one can explore which properties, if any, do have in common the possible transfer schemes that are used to implement this choice rule. While there is a subtle but important connection between these two issues, most papers in the mechanism design literature address them separately. The first question has been treated by Roberts [10], and more recently by Bikhchandani et al [2], Lavi et al [8], and Saks and Yu [11]. We note that a central assumption in all these papers exploring the characterization of implementable SCRs is that the set of social alternatives is finite. The second question, known in the literature as the Revenue Equivalence (or Payoff Equivalence) Principle, has been addressed by Green and Laffont [6], and more recently by Milgrom and Segal [9], Chung and Olszewski [4], Heydenreich et al [7] and Carbajal [3], among others.

In this paper, we attempt a somewhat more unified approach. That is, we aim to study the characterization of dominant strategy mechanisms in social choice settings with quasi-linear preferences, private valuations, social choice sets that are not necessarily finite, and rich, but not unrestricted domains of preferences over social alternatives. This characterization is composed of two elements: (i) we show that a social choice rule is dominant strategy implementable if and only if it is the solution to a well defined social optimization problem; (ii) we show that the transfer scheme that implements a corresponding SCR is unique, up to affine transformations. Thus, our result puts limits on both what (allocative) goals are attainable in this type of social choice frameworks, and what payment schemes are useful to implement these social goals. Here we are following the approach of Roberts [10], who provided a characterization of dominant strategy mechanisms with finite choice sets and multi-dimensional type spaces. Roberts showed that if the set of social alternatives consists of, say, p different allocations, and if the domain of preferences is unrestricted (i.e., the individual type space consists of the entire p-dimensional Euclidean space), then any SCR satisfying a monotonicity condition called *positive association of differences* (PAD) is implementable by generalized

Groves transfers. Since PAD is also necessary for dominant strategy implementation, it follows that with unrestricted domains and finite choice sets, a SCR is dominant strategy implementable if, and only if, it satisfies PAD. This is equivalent to the assertion that a SCR is dominant strategy implementable, if and only if, it is the solution to an affine maximization problem. Since any transfer scheme that implements an affine maximizer choice rule must be a generalized Groves transfer scheme, it follows that in this setup, any dominant strategy incentive compatible mechanism must be a generalized Groves mechanism.

The unrestricted domain assumption, which is key in Roberts [10], precludes the absence of free disposal and allocative externalities, and thus it may seem as inappropriate for economic environments with private goods. Consider for example, as Bikhchandani et al [2] do, a multi-object auction where the choice set is composed of all possible allocations of objects among the bidders. In this case, it may be natural to assume that an agent has a partial order on the set of alternatives, in the sense that if bundle x contains bundle y, then an agent will weakly prefer x to y. Bikhchandani et al showed that if one restricts the domain of types to include all types that are consistent with the initial weak order, then a condition termed *weak monotonicity* (W-MON) is sufficient and necessary for dominant strategy implementation. The main difference between Roberts [10] and Bikhchandani et al [2] is that the latter paper considers the individual type space to be a strict subset of the p-dimensional Euclidean space (where the natural number p refers again to the number of alternatives in the choice set). In this case, there may be social choice rules that satisfy PAD but not W-MON, and thus these choice rules are not implementable.

The order-based type domains of Bikhchandani et al [2] effectively reduce the heterogeneity of preferences in the mechanism design problem. If alternative x is deemed to be preferred to y according to the initial weak order, then for any type in the agent's type space, the valuation for x must be greater than or equal to the valuation for y. This restriction may be appropriate for auction-like environments, but it is not satisfactory for environments with costly disposal, allocative externalities, and little prior information regarding preference heterogeneity among agents. These features, on the other hand, are commonly found in social choice environments with public goods. Moreover, one can think of a myriad of interesting economic problems where it may be more suitable to model the set of alternatives as a continuum, or other infinite set. As illustration, consider the following examples: the provision of a public good (or multiple public projects) of variable size; the allocation of water or other divisible commodity among different populations; the design of carbon emission or, more generally, pollution permits; the location, size and shape of a wild-life reserve in certain geographic area; and so forth.

Our model is mainly motivated by this kind of problems. We consider social choice settings with quasi-linear utilities, private valuations, and a social choice set that is a metric space. We allow for costly disposal and allocative externalities (but informational externalities are ruled out). We drop the unrestricted domain assumption on preferences for social alternatives of Roberts and consider only continuous domains. That is, we model the type space of an agent as the space of continuous, real-valued functions defined on the set of social alternatives (Section 2 contains the formal description of our framework). Thus, the main differences between Roberts [10] and our paper is that we deal with fairly arbitrary (and possibly large) choice sets and continuous (not unrestricted) valuations. Within this framework, we first show in Section 3 that any dominant strategy SCR satisfies a weak monotonicity condition, which we state in terms of an individual positive association of differences. This property of a choice rule implies the collective PAD property. We proceed to show that in continuous domains, PAD is a sufficient condition for what we term *negative unanimity*, which in turn implies that the SCR is the solution to an affine maximization problem, where the social objective function is a weighted sum of valuations plus an adjustment function. We interpret this problem as one that considers each agent as a potential dictator with certain probability; thus, we conclude that any dominant strategy SCR must be a random dictatorship. Since any randomly dictatorial SCR is dominant strategy implementable, in our setting both conditions are equivalent. This equivalence constitutes the first part of our characterization result for dominant strategy mechanisms. To complete this characterization, we adapt a well known result by Green and Laffont [6] to show that in continuous domains, any transfer scheme that implements a dominant strategy SCR is a generalized Groves transfer scheme. Thus, in these domains, generalized Groves mechanisms characterize dominant strategy mechanisms.

Both our results and our exposition – although not the technical details – bear a resemblance to the Gibbard-Satterthwaite impossibility theorem as exposed by Barberà and Peleg [1]. They considered a social choice framework with a metric space as the set of social outcomes, and continuous utility functions as the domain of preferences (there is no transferable commodity in their framework). They found that any strategy-proof social choice function (SCF) is *unanimous*, i.e., if there is a social outcome that is most preferred by all agents, then this outcome will be selected by the SCF. Strategy-proofness is then shown to imply a *strong positive association* condition and a negative unanimity condition, which are then used to infer that the SCF must be dictatorial. Comparison to this result sheds some light into the role of the quasi-linearity assumption on preferences and the presence of the divisible commodity. First, in quasi-linear environments, a dominant strategy SCR need not be unanimous; this of course is due to the availability of transfers to agents to compensate for the valuation of social alternatives. Second, the strong positive association condition has to be modify to take into account differences of valuations.[1] This modification allows us to obtain the negative unanimity property, which is then used to show that a dominant strategy SCR must be a random dictatorship. Thus, with continuous preferences over large choice sets, quasi-linearity of preferences expands the type of choice rules that a society can use to make collective decisions.

[1] It is easy to speculate that this is, indeed, the reason why Roberts [10] termed his condition *positive association of differences*.

From our analysis, it is clear that one key component to this expansion is the availability of unrestricted transfers. We explore this topic in Section 4, where we show any restriction imposed on total net transfers to society creates a reversion to a dictatorship, regardless of the SCR initially considered, whenever one restricts the family of generalized Groves transfers to those satisfying a seemingly innocuous condition. To be more precise, given a randomly dictatorial choice rule, we consider the net total transfers to society as the sum of transfers to the agents plus the value of the auxiliary function used to specify the social choice problem (e.g., the cost of providing a public good). We define a budgetary restriction as a limit to the net total transfers; we then show that no dominant strategy SCR that has two or more potential dictators satisfies any budgetary restriction, if transfers are such that any gains or loses generated by a uniform shift in the valuation profile is passed along to society. The immediate corollary is, of course, that only a dictatorial SCR satisfies any budgetary restriction.

2 Preliminaries

The formal details of our social choice setting are presented in this section. There is a set $\mathcal{A}$ of social choices or alternatives, which we refer to as the *choice set*, and a finite group of agents $\mathcal{N} = \{1, \ldots, n\}$, each of whom has quasi-linear preferences over social alternatives and a divisible commodity (money). We deal with fairly general choice sets, requiring only that $\mathcal{A}$ be a compact metric space, and we denote its metric by d. Given a social alternative $x \in \mathcal{A}$ and a monetary transfer $\tau_i \in \Re$, the utility of agent $i \in \mathcal{N}$ is represented by:

$$u_i \;=\; v_i(x) \;+\; \tau_i,$$

where the individual valuation function $v_i : \mathcal{A} \to \Re$ is private information. Agent i's *type space* (valuation space) is denoted by $\mathcal{V}_i$. As usual, we let $\mathcal{V} = \times \mathcal{V}_i$ and $\mathcal{V}_{-i} = \times_{j \neq i} \mathcal{V}_j$. A *social choice environment* $\mathcal{E}$ is the tuple $\{\mathcal{N}, \mathcal{A}, \mathcal{V}\}$. This constitutes the basic framework of our analysis. We address the implementation issue by means of direct mechanisms. A direct mechanism is a pair $\Gamma = (f, t)$, where the *social choice rule* (SCR) f maps the set of types $\mathcal{V}$ into $\mathcal{A}$, and the *transfer scheme* $t = (t_1, \ldots, t_n)$ maps $\mathcal{V}$ into $\Re^n$. Our focus is on dominant strategy direct mechanisms.

Definition 1. *A direct mechanism $\Gamma = (f, t)$ is said to be dominant strategy (incentive compatible) if for every agent in the economy truth-telling is a dominant strategy; i.e., for all $i \in \mathcal{N}$, all $v_{-i} \in \mathcal{V}_{-i}$, and all $v_i, v_i' \in \mathcal{V}_i$:*

$$v_i(f(v_i, v_{-i})) \;+\; t_i(v_i, v_{-i}) \;\;\geq\;\; v_i(f(v_i', v_{-i})) \;+\; t_i(v_i', v_{-i}).$$

A SCR f is said to be dominant strategy implementable if there exists a transfer scheme $t : \mathcal{V} \to \Re^n$ such that the direct mechanism $\Gamma = (f, t)$ is dominant strategy.

Throughout this paper, we consider choice rules that are onto; i.e., given $f : \mathcal{V} \to \mathcal{A}$, it is assumed that for every social choice $x \in \mathcal{A}$ there exists a type profile $v = (v_1, \ldots, v_n) \in \mathcal{V}$ such that $x = f(v)$. One shall not consider a priori that every SCR is implementable in dominant strategies. In fact, as we will demonstrate, if the domain of individual preferences is sufficiently rich, the only implementable choice rules are solutions to a well-defined social problem that considers each agent in the economy as a potential dictator. These choice rules, also known in the literature as affine maximizers, are defined next.

Definition 2. *A social choice rule $f : \mathcal{V} \to \mathcal{A}$ is said to be a random dictatorship if there exist a vector of individual weights $k = (k_1, \ldots, k_n)$, with $k_i \in [0, 1]$ for all $i = 1, \ldots, n$ and $\sum_i^n k_i = 1$, and a function $Q : \mathcal{A} \to \Re$, such that for each type profile $v \in \mathcal{V}$:*

$$f(v) \quad \in \quad \arg\max_{a \in \mathcal{A}} \left\{ \sum\nolimits_{i=1}^{n} k_i \, v_i(a) \; + \; Q(a) \right\}.$$

We shall note that if (k, Q) is such that $k_i = \frac{1}{n}$ for all $i \in \mathcal{N}$, and $Q(x) = 0$ for all $x \in \mathcal{A}$, then a standard Groves transfer scheme can be used to implement f in dominant strategies. That is, for a SCR that maximizes social welfare, we can consider transfers $t : \mathcal{V} \to \Re^n$ to be such that for every agent i and every reported type profile $v \in \mathcal{V}$, we have $t_i(v) \; = \; \sum_{j \neq i} v_j(f(v)) \; - \; h_i(v_{-i})$. More generally, a random dictatorship can be implemented by means of generalized Groves transfers.

Definition 3. *Suppose that the SCR $f : \mathcal{V} \to \mathcal{A}$ is a random dictatorship with respect to the pair (k, Q). A transfer scheme $t : \mathcal{V} \to \Re^n$ is called generalized Groves in case the following holds:*

1. *For each agent $i \in \mathcal{N}$ with $k_i > 0$, transfers are given by*

$$t_i(v) \quad = \quad \frac{1}{k_i} \left\{ \sum\nolimits_{j \neq i} k_j \, v_j(f(v)) \; + \; Q(f(v)) \right\} \; - \; h_i(v_{-i}), \qquad \forall \, v \in \mathcal{V},$$

   where h_i is a real-valued function defined on $\mathcal{V}_{-i}$.

2. *For each agent $i \in \mathcal{N}$ with $k_i = 0$, transfers given by $t_i(v) \; = \; 0$, for all $v \in \mathcal{V}$.*

In the preceding definition, we specify payments to an agent j with $k_j = 0$ to be equal zero for any reported valuation profile, since the choices made under the SCR f will not depend on j's reports. Thus, generalized Groves transfers follow the principle of no taxation without representation.

Let $\mathcal{C}[\mathcal{A}]$ denote the space of continuous real-valued functions defined on the metric space $\mathcal{A}$. We shall use the following definition of continuous domains for individual preferences over social alternatives.

Definition 4. *The domain of preferences in a social choice environment $\mathcal{E} = \{\mathcal{N}, \mathcal{A}, \mathcal{V}\}$ is said to be continuous if the choice set $\mathcal{A}$ is a compact metric space, and if it is the case that $\mathcal{V}_i = \mathcal{C}[\mathcal{A}]$, for every agent $i \in \mathcal{N}$.*

Thus, with continuous domain of preferences, the space of types is given by $\mathcal{V} = (\mathcal{C}[\mathcal{A}])^n$. Observe that this setup allows for large social choice sets (e.g., $\mathcal{A} = [0, M]$, etc.), costly disposal, and allocative externalities, but the domain of preferences over social alternatives is not unrestricted. Before proceeding with the characterization of dominant strategy mechanisms, we explore some implications of the continuous domain assumption.

Lemma 5. *Suppose $\mathcal{V}_i = \mathcal{C}[\mathcal{A}]$ and let $x, y \in \mathcal{A}$, with $x \neq y$.*

1. *For any $v_i \in \mathcal{V}_i$ and any real number $\epsilon_i > 0$, there exists a valuation function $\hat{v}_i \in \mathcal{V}_i$ satisfying $\hat{v}_i(x) = v_i(x)$, $\hat{v}_i(y) = v_i(y) - \epsilon_i$, and $\hat{v}_i(x) - v_i(x) > \hat{v}_i(z) - v_i(z)$, for all $a \in \mathcal{A}$, $a \neq x$.*

2. *Let v_i, v_i' be any two valuations in $\mathcal{V}_i$ for which $v_i'(x) - v_i(x) > v_i'(y) - v_i(y)$. Then there exists $\hat{v}_i \in \mathcal{V}_i$ such that*

$$\begin{aligned} \hat{v}_i(x) - v_i(x) &> \hat{v}_i(a) - v_i(a), && \forall\, a \in \mathcal{A},\, a \neq x, \\ \hat{v}_i(y) - v_i'(y) &> \hat{v}_i(a) - v_i'(a), && \forall\, a \in \mathcal{A},\, a \neq y. \end{aligned}$$

Proof 5.1. Define the function $\hat{v}_i$ on $\mathcal{A}$ by $\hat{v}_i(a) = v_i(a) - \epsilon_i \frac{d(a,x)}{d(y,x)}$, for each $a \in \mathcal{A}$. Of course $d(y, x) > 0$, since $y \neq x$, and thus $\hat{v}_i \in \mathcal{V}_i$. Then, clearly we have $\hat{v}_i(x) = v_i(x)$, $\hat{v}_i(y) = v_i(y) - \epsilon_i$, and further $\hat{v}_i(a) - v_i(a) = -\epsilon_i \frac{d(a,x)}{d(y,x)} < 0$, for all $a \neq x$.

5.2. Let $v_i, v_i' \in \mathcal{V}_i$ be such that $v_i'(x) - v_i(x) > v_i'(y) - v_i(y)$. Choose $\epsilon_i, \delta_i \in \Re$ to satisfy $\epsilon_i = \{v_i'(x) - v_i(x)\} - \{v_i'(y) - v_i(y)\} > 0$, and $\delta_i = v_i'(x) - v_i(x) - \frac{1}{2}\epsilon_i = v_i'(y) - v_i(y) + \frac{1}{2}\epsilon_i$. Now we define the function $\hat{v}_i$ on $\mathcal{A}$ as follows: for every $a \in \mathcal{A}$,

$$\hat{v}_i(a) = \min\Big\{v_i(a) - \frac{\epsilon_i}{2}\frac{d(a,x)}{d(y,x)},\ v_i'(a) - \frac{\epsilon_i}{2}\frac{d(a,y)}{d(y,x)} - \delta_i\Big\}.$$

We observe first that $\hat{v}_i \in \mathcal{V}_i$. Further, we notice that

$$\begin{aligned} \hat{v}_i(x) - v_i(x) &= \min\{0\,,\, v_i'(x) - v_i(x) - \frac{\epsilon_i}{2} - \delta_i\} = 0, \\ \hat{v}_i(y) - v_i'(y) &= \min\{v_i(y) - v_i'(y) - \frac{\epsilon_i}{2}\,,\, -\delta_i\} = -\delta_i. \end{aligned}$$

The result now follows from the fact that $\hat{v}_i(a) - v_i(a) \leq -\frac{\epsilon_i}{2}\frac{d(a,x)}{d(y,x)} < 0$, for all $a \in \mathcal{A}$, $a \neq x$, and similarly $\hat{v}_i(a) - v_i'(a) \leq -\frac{\epsilon_i}{2}\frac{d(a,y)}{d(y,x)} - \delta_i < -\delta_i$, for all $a \in \mathcal{A}$, $a \neq y$. ■

3 Characterization of dominant strategy mechanisms

We state the main result of our paper, which puts limits on both the class of SCRs that can be implemented in dominant strategies in environments with quasi-linear utilities and continuous preferences for social alternatives, and on the class of transfers schemes that can be used to accomplish implementation.

Theorem 6. *Let $\mathcal{E} = \{\mathcal{N}, \mathcal{A}, \mathcal{V}\}$ be a social choice environment satisfying the continuous domain assumption. Let $\Gamma = (f, t)$ be a direct mechanism with an onto choice rule. Then Γ is dominant strategy incentive compatible if, and only if, its SCR is a random dictatorship and its transfer scheme is generalized Groves.*

We divide the proof of Theorem 6 in several propositions. Following Roberts [10], we start by showing that dominant strategy implementation of a SCR implies a monotonicity condition which we term *individual positive association of differences* (this condition is similar to the *weak monotonicity* condition of Bikhchandani et al [2]). This, in turn, implies the *positive association of difference* (PAD) condition (Lemma 8 and Proposition 9). We then prove that if a choice rule satisfies PAD, then it must satisfy a *negative unanimity* condition similar to one that appears in the proof of the Gibbard-Satterthwaite theorem (see Lemma 10; compare it to Lemma 4.9 of Barberà and Peleg [1]). This parallel is important in that it highlights the similarities and differences between the result with general continuous utility functions and arbitrary metric spaces, as presented in Barberà and Peleg [1] and the result in the case of continuous, quasi-linear utility functions. In both cases, the social choice function must satisfy negative unanimity but only in the general case it must satisfy *unanimity.* This follows since in the quasi-linear case, transfers can be used to select an alternative that is not the most preferred one for all individuals.

With these results at hand, we show that a dominant strategy implementable SCR must be randomly dictatorial with respect to some pair (k, Q); this is presented in Proposition 14. To accomplish this result, we associate to every pair of distinct social alternatives (x, y) a particular set in $\Re^n$, the Euclidean space where the vectors of valuation differences lie. We then employ PAD and the negative unanimity property to show that, in the case of continuous preference domains, these sets are equivalent after a certain normalization has been imposed on them. This normalization gives us the weights $(k_1, \ldots, k_n)$ and the external contribution function Q needed to construct the affine maximization problem whose solution is a random dictatorship (see Lemmas 11 to 13). This clever technique was first presented by Roberts [10], and later refined by Lavi et al [8], for the case of finite choice sets and unrestricted domains. The general structure of our proof of Proposition 14 builds upon the exposition of Lavi et al. Our task is however complicated by the fact that we deal with choice sets that are arbitrary metric spaces and preferences on social alternatives that are continuous functions (see especially Lemma 11).

The next step is easily established: a randomly dictatorial SCR is dominant strategy implementable by generalized Groves transfers (Proposition 15) Finally, we show that any transfer scheme that implements a SCR in dominant strategies must be a generalized Groves transfer scheme (see Proposition 16); for this one can adapt the argument originally presented by Green and Laffont [6]. These two last results are well-known in the literature and we include them here for completeness. We shall note that this implies as a corollary the important Revenue Equivalence principle: in continuous domains, any two dominant strategy mechanisms sharing the same social choice rule generate transfers that are equal, up to an affine transformation.

For the remaining of the paper, we employ the following notation: for any $a \in \mathcal{A}$, denote $v(a) = (v_1(a), \ldots, v_n(a))$, and write $v(x) \gg v(y)$ if it is the case that $v_i(x) > v_i(y)$ for each agent $i \in \mathcal{N}$, for $x \neq y$. Both individual positive association of differences and the standard positive association of differences are defined next.

Definition 7. *The SCR $f : \mathcal{V} \to \mathcal{A}$ satisfies the individual positive association of differences condition (i–PAD) if for any $i \in \mathcal{N}$, any type profile $v \in \mathcal{V}$, and any $\hat{v}_i \in \mathcal{V}_i$, the following holds: $x = f(v_i, v_{-i})$ and $\left[\hat{v}_i(x) - v_i(x) > \hat{v}_i(a) - v_i(a), \text{ all } a \in \mathcal{A}, a \neq x\right]$ imply that $x = f(\hat{v}_i, v_{-i})$.*

The SCR $f : \mathcal{V} \to \mathcal{A}$ satisfies the collective positive association of differences condition (PAD) if for any two type profiles $v, \hat{v} \in \mathcal{V}$, the following holds: $x = f(v)$ and $\left[\hat{v}(x) - v(x) \gg \hat{v}(a) - v(a), \text{ all } a \in \mathcal{A}, a \neq x\right]$ imply that $x = f(\hat{v})$.

We next show that i–PAD is a necessary condition for dominant strategy implementation of SCRs; since i–PAD implies PAD, it follows that PAD is a necessary condition for implementation of the choice rule f.

Lemma 8. *If $f : \mathcal{V} \to \mathcal{A}$ is dominant strategy implementable, then it satisfies i–PAD.*

Proof Suppose the contrary, and let $v \in \mathcal{V}$ and $\hat{v}_i \in \mathcal{V}_i$ be such that $f(v_i, v_{-i}) = x$ and $\left[\hat{v}_i(x) - v_i(x) > \hat{v}_i(a) - v_i(a), \forall a \neq x\right]$, but instead assume that $f(\hat{v}_i, v_{-i}) = y$, for some $y \in \mathcal{A}$, $y \neq x$. The implementability of the SCR f now implies that there exists a transfer scheme $t : \mathcal{V} \to \Re^n$ for which the following holds:

$$\begin{aligned} v_i(x) + t_i(v_i, v_{-i}) &\geq v_i(y) + t_i(\hat{v}_i, v_{-i}) \qquad \text{and} \\ \hat{v}_i(y) + t_i(\hat{v}_i, v_{-i}) &\geq \hat{v}_i(x) + t_i(v_i, v_{-i}). \end{aligned}$$

These two expressions yield to $v_i(x) - v_i(y) \geq t_i(\hat{v}_i, v_{-i}) - t_i(v_i, v_{-i}) \geq \hat{v}_i(x) - \hat{v}_i(y)$.[2] Rearranging, we obtain $\hat{v}_i(x) - v_i(x) \leq \hat{v}_i(y) - v_i(y)$, which is a contradiction to our initial assumption. ■

[2]This is similar to the property that has been called weak monotonicity(W-MON) by Bikhchandani et al [2]. They showed that with finite choice sets and restricted domains, W-MON is sufficient and necessary for implementation. See the discussion given in Section 1.

Proposition 9. *If the choice rule $f : \mathcal{V} \to \mathcal{A}$ is dominant strategy implementable, then it satisfies PAD.*

Proof In light of Lemma 8, it suffices to show that i–PAD implies PAD. Thus, let $f : \mathcal{V} \to \mathcal{A}$ be a SCR satisfying i–PAD. Let $v, \hat{v} \in \mathcal{V}$ be two type profiles such that $\hat{v}(x) - v(x) \gg \hat{v}(a) - v(a)$, for all $a \in \mathcal{A}$, $a \neq x$, and suppose that $f(v) = x$.

Write $v^0 = v$ and for $i = 1, \ldots, n$, define $v^i = (\hat{v}_1, \ldots, \hat{v}_i, v_{i+1}, \ldots, v_n)$, so that $v^1 = (\hat{v}_1, v_2, \ldots v_n)$ and $v^n = \hat{v}$. Then we have $x = \psi(v^0)$, and it is immediate to see that i–PAD implies $x = f(v^1)$. Now notice that for any $i = 1, \ldots, n$, $x = f(v^{i-1})$ and $[v_i^i(x) - v_i^{i-1}(x) > v_i^i(z) - v_i^{i-1}(z),\ \forall\, z \neq x]$ imply $x = f(v^i)$. The result follows now by induction. ■

From Proposition 9 one infers that PAD is a necessary condition for dominant strategy implementation. It shall be noted that this result holds for any social choice setting $\mathcal{E}$, not just environments that satisfy the continuous domain assumption. On the other hand, Proposition 14, which shows that if the choice rule f satisfies PAD, then it is a randomly dictatorial SCR, makes extensive use of the continuous domain assumption. We arrive at this conclusion using Lemmas 10 through 13. The first lemma of the trio states that PAD is a sufficient condition to obtain *negative unanimity.* This property is useful in that it allows one to discard social alternatives as potential selections.

Lemma 10. *Let $f : \mathcal{V} \to \mathcal{A}$ be a choice rule satisfying PAD. For any two valuation profiles $v, v' \in \mathcal{V}$, if it is the case that $x = f(v)$ and $v'(x) - v(x) \gg v'(y) - v(y)$, for some $y \in \mathcal{A}$, $y \neq x$, then it must be that $y \neq f(v')$.*

Proof Suppose to obtain a contradiction that $y = f(v')$. Using Lemma 5.2, we find that there exists a valuation profile $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \in \mathcal{V}$ such that $\hat{v}(x) - v(x) \gg \hat{v}(a) - v(a)$, all $a \neq x$, and further $\hat{v}(y) - v'(y) \gg \hat{v}(a) - v'(a)$, all $a \neq y$. Since the choice rule f satisfies PAD, $x = f(v)$ implies $x = f(\hat{v})$, and similarly $y = f(v')$ implies $y = f(\hat{v})$. Since $x \neq y$, this is impossible. Therefore, we conclude that $y \neq f(v')$. ■

We now construct the appropriate sets in $\Re^n$ associated with each pair of distinct social alternatives (x, y). In what follows, fix an onto choice rule f, so that for every alternative $x \in \mathcal{A}$, there exists a type profile $v \in \mathcal{V}$ such that $x = f(v)$. Take any alternative $y \in \mathcal{A}$, $y \neq x$, and write the valuation difference as $v(x) - v(y) = (\alpha_1, \ldots, \alpha_n) = \alpha$. For this SCR, we define the set $H(x, y) \subseteq \Re^n$ by

$$H(x,y) \ := \ \{\alpha \in \Re^n \mid \exists\, v \in \mathcal{V} \text{ such that } x = f(v) \text{ and } \ v(x) - v(y) = \alpha\}$$

Observe that, by construction, $H(x, y)$ is a non-empty subset of $\Re^n$. Let P_α denote the strictly positive orthant translated by α; i.e., $P_\alpha = \{\alpha + \Re^n_{++}\}$. It will be seen that if α belongs to $H(x, y)$, then P_α is a subset of $H(x, y)$. This allows us to derive important properties of the sets $H(x, y)$.

Lemma 11. *Let $f : \mathcal{V} \to \mathcal{A}$ satisfy PAD. Let x, y be two alternatives in $\mathcal{A}$, with $x \neq y$, and suppose that $\alpha \in H(x, y)$. The following properties hold:*

1. $P_\alpha \cap H(x, y) \;=\; P_\alpha$ *and* $-P_\alpha \cap H(y, x) \;=\; \emptyset$.

2. *If* $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \Re^n$, $\epsilon \gg \mathbf{0}$, *and* $\alpha - \epsilon \notin H(x, y)$, *then* $-(\alpha - \epsilon) \in H(y, x)$.

In addition, let $z \in \mathcal{A}$, $z \neq x, y$, and assume $\alpha' \in H(y, z)$. Then the following holds:

3. $P_{\alpha+\alpha'} \subseteq H(x, z)$.

Proof 11.1. Let $\alpha \in H(x, y)$, so that for some type profile $v \in \mathcal{V}$, we have $x = f(v)$ and $v(x) - v(y) \;=\; \alpha$. Let ϵ be any vector in $\Re^n$ satisfying $\epsilon \gg \mathbf{0}$. We shall see that $\beta \;=\; \alpha + \epsilon$ belongs to $H(x, y)$. From Lemma 5.1, we infer the existence of a profile $\hat{v} \in \mathcal{V}$ such that $\hat{v}(x) - v(x) \gg \hat{v}(a) - v(a)$, for all $a \in \mathcal{A}$, $a \neq x$, with $\hat{v}(x) - v(x) \;=\; \hat{v}(y) - v(y) + \epsilon$. Since f satisfies PAD. Thus, it follows that $x = f(\hat{v})$, and of course $\hat{v}(x) - \hat{v}(y) \;=\; \alpha + \epsilon \;=\; \beta$. Thus $\beta \in H(x, y)$, as claimed.

To obtain the second implication, suppose by contradiction that $-\beta \;=\; -(\alpha + \epsilon)$ belongs to $H(y, x)$, for some $\epsilon \gg \mathbf{0}$. Since $\alpha \in H(x, y)$, there exists a profile of types $v \in \mathcal{V}$ such that $x \;=\; f(v)$ and $v(x) - v(y) \;=\; \alpha$. Similarly, since $-\beta \in H(y, x)$, there exists a profile of types $\hat{v} \in \mathcal{V}$ such that $y \;=\; f(\hat{v})$ and $\hat{v}(y) - \hat{v}(x) \;=\; -\beta$. Notice that we have $\beta \;=\; \hat{v}(x) - \hat{v}(y) \gg v(x) - v(y) \;=\; \alpha$, therefore $\hat{v}(x) - v(x) \gg \hat{v}(y) - v(y)$. Using the negative unanimity property of Lemma 10, $x = f(v)$ implies that $y \neq f(\hat{v})$, which is a contradiction.

11.2. Let $\alpha \in H(x, y)$, so that there exists $v \in \mathcal{V}$ such that $x = f(v)$ and $v(x) - v(y) \;=\; \alpha$. Let $\epsilon \gg \mathbf{0}$ and assume that $\alpha - \epsilon \notin H(x, y)$. Suppose to obtain a contradiction that $-(\alpha - \epsilon) \notin H(y, x)$. We construct the profile of functions $v' \;=\; (v'_1, \ldots, v'_n)$ as follows. For each $i \in \mathcal{N}$, define v'_i on $\mathcal{A}$ by:

$$v'_i(a) \;\;=\;\; v_i(a) \;+\; \left(\frac{1}{2} \;-\; \frac{d(a, y)}{d(x, y)}\right)\epsilon_i$$

Observe that the function profile v' belongs to $\mathcal{V}$, by the rich domain assumption. Further, we notice that $v'(x) \;=\; v(x) - \frac{1}{2}\epsilon$, $\;v'(y) \;=\; v(y) + \frac{1}{2}\epsilon$, and thus we have that $v'(x) - v'(y) \;=\; v(x) - v(y) - \epsilon \;=\; \alpha - \epsilon$. It follows that $x \neq f(v')$, for otherwise $\alpha - \epsilon \in H(x, y)$. Similarly, $y \neq f(v')$, for otherwise $v'(y) - v'(x) \;=\; -(\alpha - \epsilon) \in H(y, x)$, which is contrary to our previous assumption. We conclude that there exists $z \in \mathcal{A}$, $z \neq x, y$, such that $z = f(v')$. Without loss of generality, we assume that $d(z, y) > d(x, y)$. Notice that we can write:

$$\begin{aligned}\{v'(x) \;-\; v(x)\} \;-\; \{v'(z) \;-\; v(z)\} \;\;&=\;\; -\frac{1}{2}\epsilon \;-\; \frac{1}{2}\epsilon \;+\; \frac{d(z, y)}{d(x, y)}\epsilon \\ &=\;\; \left(\frac{d(z, y)}{d(x, y)} \;-\; 1\right)\epsilon \;\;\gg\;\; \mathbf{0}.\end{aligned}$$

Thus, we conclude that there exists a valuation profile $v' \in \mathcal{V}$ satisfying $z = f(v')$ and $v'(x) - v(x) \gg v'(z) - v(z)$. Using Lemma 5.2, we argue that there exists a type profile $\hat{v} \in \mathcal{V}$ such that

$$\hat{v}(x) - v(x) \gg \hat{v}(a) - v(a), \qquad \forall\, a \neq x, \tag{1}$$

$$\hat{v}(z) - v'(z) \gg \hat{v}(a) - v'(a), \qquad \forall\, a \neq z. \tag{2}$$

Since $z = f(v')$, expression (2) and PAD imply that $z = f(\hat{v})$. We now use (1) to write $v(z) - \hat{v}(z) \gg v(x) - \hat{v}(x)$. Since f satisfies the negative unanimity property, this last implies that $x \neq f(v)$, which is a contradiction.

11.3. Let d^* be a real number satisfying $0 < d^* < \frac{1}{2}\inf\{d(x,y), d(y,z), d(x,z)\}$, and denote the closed balls of radius d^* around x, y and z, respectively, by $B_x = \{a \in \mathcal{A} : d(a,x) \leq d^*\}$, $B_y = \{a \in \mathcal{A} : d(a,y) \leq d^*\}$, and $B_z = \{a \in \mathcal{A} : d(a,z) \leq d^*\}$, with $B_x^c = \mathcal{A} - B_x$, $B_y^c = \mathcal{A} - B_y$, and $B_z^c = \mathcal{A} - B_z$.

Since $\alpha \in H(x,y)$, there exists a valuation profile $v \in \mathcal{V}$ such that $x = f(v)$ and $v(x) - v(y) = \alpha$. Similarly, since $\alpha' \in H(y,z)$, there exists a type profile $v' \in \mathcal{V}$ such that $y = f(v')$ and $v'(y) - v'(z) = \alpha'$. Without loss of generality, we assume that v and v' are such that for all $i \in \mathcal{N}$, for all $a \in B_z$, one has $v_i'(a) - v_i(a) > v_i'(y) - v_i(y)$. Let $\delta \in \Re^n$ be such that $\delta = v'(y) - v(y)$. Define the function profile $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)$ as follows: for each $i = 1, \ldots, n$, for all $a \in \mathcal{A}$:

$$\begin{aligned} \hat{v}_i(a) \;=\; & v_i(a) + \frac{d(a, B_x^c)}{d^*}\,\epsilon_i + \frac{d(a, B_y^c)}{d^*}\frac{\epsilon_i}{2} + \frac{d(a, B_z^c)}{d^*}\left[v_i'(a) - v_i(a)\right] \\ & + \left[1 - \frac{d(a, B_z^c)}{d^*}\right]\delta_i. \end{aligned}$$

It follows that $\hat{v}_i \in \mathcal{C}[\mathcal{A}]$, all $i \in \mathcal{N}$, and thus $\hat{v} \in \mathcal{V}$. Now observe that $\hat{v}_i(x) = v_i(x) + \epsilon_i + \delta_i$, while for all $a \in B_x, a \neq x$, one has $\hat{v}_i(a) = v_i(a) + t\epsilon_i + \delta_i$, for $0 \leq t < 1$. Note also that if $a \in B_y$, then $\hat{v}_i(a) = v_i(a) + \frac{t'}{2}\epsilon_i + \delta_i$, for $0 \leq t' \leq 1$; if $a \in B_z$, then $\hat{v}_i(a) = v_i(a) + t'[v_i'(a) - v_i(a)] + (1-t')\delta_i$, where $0 \leq t' \leq 1$; while if $a \in B_x^c \cap B_y^c \cap B_z^c$, we have that $\hat{v}_i(a) = v_i(a) + \delta_i$. These possibilities are exhaustive, for any $a \in \mathcal{A}$. The following then holds:

$$\forall\, a \in B_x,\, a \neq x, \quad \hat{v}(a) - v(a) = t\,\epsilon + \delta \qquad (0 \leq t < 1); \tag{3}$$

$$\forall\, a \in B_y, \quad \hat{v}(a) - v(a) = \frac{t'}{2}\,\epsilon + \delta \qquad (0 \leq t' \leq 1); \tag{4}$$

$$\forall\, a \in B_x^c \cap B_y^c \cap B_z^c, \quad \hat{v}(a) - v(a) = \delta. \tag{5}$$

Expressions (3) to (5), together with $\hat{v}(x) - v(x) = \epsilon + \delta$, imply that for every $a \in B_z^c$, $a \neq x$, we have $\hat{v}(x) - v(x) \gg \hat{v}(a) - v(a)$. Since $x = f(v)$, the negative unanimity property of Lemma 10 now yields to $a \neq f(\hat{v})$, for any $a \in \mathcal{A} - B_z, a \neq x$.

Let $a \in B_z$ instead. Then we have that for some $0 \leq t' \leq 1$:

$$\hat{v}(y) - \hat{v}(a) = \Big\{v(y) + \frac{1}{2}\epsilon + \delta\Big\} - \Big\{v(a) + t'\big[v'(a) - v(a)\big] + (1-t')\delta\Big\}.$$

We replace $\delta = v'(y) - v(y)$ in the above equation, add and subtract $v'(a)$ to obtain

$$\hat{v}(y) - \hat{v}(a) = v'(y) - v'(a) + (1-t')\big\{v'(a) - v(a) - \delta\big\} + \frac{1}{2}\epsilon.$$

Hence, for any $a \in B_z$,

$$\{\hat{v}(y) - v'(y)\} - \{\hat{v}(a) - v'(a)\} = (1-t')\big[\{v'(a) - v(a)\} - \{v'(y) - v(y)\}\big] + \frac{1}{2}\epsilon.$$

Note that in the above equation, $0 \leq t' < 1$ for all $a \in B_z, a \neq z$, and $t' = 1$ for $a = z$. Since the type profiles v and v' were chosen to satisfy $v'(a) - v(a) \gg v'(y) - v(y)$, for each $a \in B_z$, it follows that in this case $\hat{v}(y) - v'(y) \gg \hat{v}(a) - v'(a)$. We use the negative unanimity property again to conclude that $y = f(v')$ now implies that $a \neq f(\hat{v})$.

Thus, we shall conclude that $x = f(\hat{v})$. Moreover,

$$\begin{aligned}\hat{v}(x) - \hat{v}(z) &= \{v(x) + \epsilon + \delta\} - v'(z)\\ &= v(x) - v(y) + v'(y) - v'(z) + \epsilon\\ &= \alpha + \alpha' + \epsilon.\end{aligned}$$

This shows that $\alpha + \alpha' + \epsilon \in H(x, z)$, as desired. ■

Lemma 11 provides the basic structure of the set $H(x, y) \subseteq \Re^n$ (see Figure 1 below). We proceed to look for a common structure for the sets $H(x, y)$ and $H(w, z)$. This common structure turns out to satisfy important properties, which allows us to construct the appropriate affine maximization problem. In the sequel, we employ the following notation: if q is a real number, let $\boldsymbol{q} = (q, \ldots, q)$ be a vector in $\Re^n$. For any $x, y \in \mathcal{A}$, $x \neq y$, we define $\underline{q}(x, y) := \inf\{q \in \Re \mid \boldsymbol{q} \in H(x, y)\}$ and denote the vector of same coordinates $(\underline{q}(x, y), \ldots, \underline{q}(x, y)) \in \Re^n$ by $\underline{\boldsymbol{q}}(x, y)$. This vector can be thought of a lower bound for the set $H(x, y)$. We shall use these lower bounds to shift the different sets $H(x, y)$ and $H(w, z)$.

Lemma 12. *Let $f : \mathcal{V} \to \mathcal{A}$ be a SCR satisfying PAD. Let x, y, z be distinct social alternatives in $\mathcal{A}$. Then the following holds:*

1. $-\infty < \underline{q}(x, y) < +\infty$.
2. $\underline{q}(x, y) = -\underline{q}(y, x)$.
3. $\underline{q}(x, z) = \underline{q}(x, y) + \underline{q}(y, z)$.

Proof 12.1. Note that the set $\{q \in \Re \mid \boldsymbol{q} \in H(x,y)\}$ is non-empty. Indeed, consider the type profile $v \in \mathcal{V}$ to be such that $x = f(v)$ and $v(x) - v(y) = \alpha \in H(x,y)$. Let $q \in \Re$ satisfy $q > \max\{\alpha_i : i = 1, \ldots, n\}$. Choose $\epsilon \in \Re^n, \epsilon \gg \mathbf{0}$, so that $\alpha_i + \epsilon_i = q$ for each $i \in \mathcal{N}$. By Lemma 11.1, we have $\boldsymbol{q} \in H(x,y)$. Since $\{q \in \Re \mid \boldsymbol{q} \in H(x,y)\}$, it follows that $\underline{q}(x,y) < +\infty$. Now suppose by contradiction that the set $H(x,y)$ is unbounded from below. Then, using Lemma 11.1 again, we conclude that $H(y,x)$ is empty, which is a contradiction. It follows that $-\infty < \underline{q}(x,y) < +\infty$, as claimed.

12.2. Fix any $\epsilon \in \Re^n$, $\epsilon \gg \mathbf{0}$; then we have that $\underline{\boldsymbol{q}}(x,y) + \frac{\epsilon}{2} \in H(x,y)$. Note that, from Lemma 11.1, writing $\underline{\boldsymbol{q}}(x,y) + \epsilon = (\underline{\boldsymbol{q}}(x,y) + \frac{\epsilon}{2}) + \frac{\epsilon}{2}$ implies that $-(\underline{\boldsymbol{q}}(x,y) + \epsilon) \notin H(y,x)$. On the other hand, since by definition $\underline{\boldsymbol{q}}(x,y) - \frac{\epsilon}{2} \notin H(x,y)$, we write $\underline{\boldsymbol{q}}(x,y) - \frac{\epsilon}{2} = (\underline{\boldsymbol{q}}(x,y) + \frac{\epsilon}{2}) - \epsilon$ and use Lemma 11.2 to conclude that $-\underline{\boldsymbol{q}}(x,y) + \frac{\epsilon}{2}$ belongs to $H(y,x)$, and so does $-\underline{\boldsymbol{q}}(x,y) + \epsilon$. We summarize the preceding argument in the following terms: for any $\epsilon \in \Re^n, \epsilon \gg \mathbf{0}$, one has $-\underline{\boldsymbol{q}}(x,y) + \epsilon \in H(y,x)$ and $-\underline{\boldsymbol{q}}(x,y) - \epsilon \notin H(y,x)$. Thus, $-\underline{q}(x,y) = \inf\{q \in \Re \mid \boldsymbol{q} \in H(y,x)\} =: \underline{q}(y,x)$, as desired.

12.3. Again, fix an arbitrary $\epsilon \in \Re^n$, $\epsilon \gg \mathbf{0}$. Denoting $\underline{\boldsymbol{q}}(x,y) = (\underline{q}(x,y), \ldots, \underline{q}(x,y)) \in \Re^n$ and $\underline{\boldsymbol{q}}(y,z) = (\underline{q}(y,z), \ldots, \underline{q}(y,z)) \in \Re^n$, we have that $\underline{\boldsymbol{q}}(x,y) + \frac{\epsilon}{4} \in H(x,y)$ and $\underline{\boldsymbol{q}}(y,z) + \frac{\epsilon}{4} \in H(y,z)$. From Lemma 11.3, it follows that $\underline{\boldsymbol{q}}(x,y) + \underline{\boldsymbol{q}}(y,z) + \epsilon \in H(x,z)$, and hence $\underline{q}(x,z) \leq \underline{q}(x,y) + \underline{q}(y,z)$. Reversing the role of x and z, one readily infers that $\underline{q}(z,x) \leq \underline{q}(z,y) + \underline{q}(y,x)$. Using 12.2, this is equivalent to $-\underline{q}(x,z) \leq -\underline{q}(x,y) - \underline{q}(y,z)$. The desired result now follows. ■

To normalize the set $H(x,y)$ using $\underline{\boldsymbol{q}}(x,y) = (\underline{q}(x,y), \ldots, \underline{q}(x,y))$, define the set $I(x,y) := H(x,y) - \underline{\boldsymbol{q}}(x,y)$, and denote its interior by $I^{\circ}(x,y)$; where x, y in $\mathcal{A}$, $x \neq y$.

Lemma 13. *For any w, x, y, z in $\mathcal{A}$, with $x \neq y$ and $w \neq z$, the following holds:*

1. *Given any $\epsilon \gg \mathbf{0}$, $\epsilon \in I^{\circ}(x,y)$ and $\mathbf{0} \notin I^{\circ}(x,y)$.*
2. *$I^{\circ}(x,y) = I^{\circ}(w,z) =: I^{\circ}$.*
3. *I° is a convex subset of $\Re^n$.*

Proof 13.1. Fix $\epsilon \in \Re^n$, $\epsilon \gg \mathbf{0}$. Then it is the case that $\underline{\boldsymbol{q}}(x,y) + \epsilon \in H(x,y)$, and thus we have $\epsilon \in I(x,y)$. To show that ϵ is indeed in the interior set $I^{\circ}(x,y)$, let t be a positive real number satisfying $t = \frac{1}{2}\inf\{\epsilon_i : i = 1, \ldots, n\}$, and define the open neighborhood around ϵ by $N_\epsilon = \{\epsilon' \in \Re^n : |\epsilon'_i - \epsilon_i| < t\}$. Notice that for each $\epsilon' \in N_\epsilon$ and each $i = 1, \ldots, n$, we have $0 < \epsilon_i - t < \epsilon'_i < \epsilon_i + t$. Thus, given any $\epsilon' \in N_\epsilon$, we have that $\underline{\boldsymbol{q}}(x,y) + \epsilon' \in H(x,y)$, and thus $\epsilon' \in I(x,y)$. Therefore, we conclude that $\epsilon \in I^{\circ}(x,y)$.

To obtain the second implication, suppose on the contrary that $\mathbf{0}$ indeed belongs to the interior of $I(x,y)$. Then, there must exist a positive real number t_0 such that the neighborhood $N_0 = \{\epsilon' \in \Re^n : |\epsilon'_i| < t_0\}$ is a subset of $I(x,y)$. Take $\epsilon'' = (\epsilon''_1, \ldots, \epsilon''_n)$ such that $0 < \epsilon''_i < t_0$ for all $i = 1, \ldots, n$. Since it is the case that $-\epsilon''$ belongs to N_0, it follows that $-\epsilon'' \in I(x,y)$. This, in turn, implies that $\underline{\boldsymbol{q}}(x,y) - \epsilon'' \in H(x,y)$, which is a contradiction.

13.2. Consider three distinct alternatives x, y, z in $\mathcal{A}$. Fix $\epsilon \in \Re^n$, $\epsilon \gg \mathbf{0}$, and let $\alpha \in H(x,y)$. Then, since $\underline{\boldsymbol{q}}(y,z) + \frac{1}{4}\epsilon$ belongs to $H(y,z)$, it follows from Lemma 11.3 that $\beta = \alpha + \underline{\boldsymbol{q}}(y,z) + \frac{1}{2}\epsilon \in H(x,z)$. Thus, we now can write:

$$\begin{aligned}(\alpha - \underline{\boldsymbol{q}}(x,y)) + \epsilon &= \beta - \underline{\boldsymbol{q}}(x,y) - \underline{\boldsymbol{q}}(y,z) + \frac{1}{2}\epsilon \\ &= \beta - \underline{\boldsymbol{q}}(x,z) + \frac{1}{2}\epsilon;\end{aligned}$$

where the last equality follows from Lemma 12.3. Observe that the left-hand side of the above equalities belongs to $I^\circ(x,y)$ and the right-hand side belongs to $I^\circ(x,z)$. Since α and ϵ are arbitrary, this shows that any element in $I^\circ(x,y)$ is also in $I^\circ(x,z)$, thus $I^\circ(x,y) \subseteq I^\circ(x,z)$. Reversing the roles of y and z, we obtain the reverse set inclusion. Thus, we have that $I^\circ(x,y) = I^\circ(x,z)$.

Now consider three distinct alternatives w, x, y in $\mathcal{A}$. Fix $\epsilon' \in \Re^n$, $\epsilon' \gg \mathbf{0}$, and let $\alpha' \in H(x,z)$. We have $\underline{\boldsymbol{q}}(w,x) + \frac{1}{4}\epsilon' \in H(w,x)$ and thus $\beta' = \alpha' + \underline{\boldsymbol{q}}(w,x) + \frac{1}{2}\epsilon' \in H(w,z)$. A reasoning similar to the one carried previously shows that any element in $I^\circ(x,z)$ belongs also to $I^\circ(w,z)$, thus $I^\circ(x,z) \subseteq I^\circ(w,z)$. Reversing the roles of x and w, we conclude that $I^\circ(x,z) = I^\circ(w,z)$. Hence, for all different alternatives w, x, y, z in $\mathcal{A}$, we have that $I^\circ(x,y) = I^\circ(x,z) = I^\circ(w,z)$. This last equation, used in suitable combinations, allows us to deduce that for $x \neq y$ and $w \neq z$, $I^\circ(x,y) = I^\circ(w,z) = I^\circ$. For instance, we can do $I^\circ(x,y) = I^\circ(x,z) = I^\circ(y,z) = I^\circ(y,x)$.

13.3. Since the set I° is open in $\Re^n$, it suffices to show that it is mid-convex (Roberts [10]). We first show that if β and β' belong to I°, then so does $\beta + \beta'$. Later, we show that $\frac{1}{2}\beta''$ is in I° provided $\beta'' \in I^\circ$. This last step will complete our argument.

Suppose that $\beta, \beta' \in I^\circ$. Then, by Lemma 13.2, we can find $\alpha \in H(x,y)$, $\alpha' \in H(y,z)$, and $\epsilon, \epsilon' \gg \mathbf{0}$ such that $\beta = \alpha - \underline{\boldsymbol{q}}(x,y) + \epsilon$ and $\beta' = \alpha' - \underline{\boldsymbol{q}}(x,y) + \epsilon'$. Notice that:

$$\begin{aligned}\beta + \beta' &= \left(\alpha + \alpha' + \frac{1}{2}(\epsilon + \epsilon')\right) - \underline{\boldsymbol{q}}(x,y) - \underline{\boldsymbol{q}}(y,z) + \frac{1}{2}(\epsilon + \epsilon') \\ &= \left(\alpha + \alpha' + \frac{1}{2}(\epsilon + \epsilon')\right) - \underline{\boldsymbol{q}}(x,z) + \frac{1}{2}(\epsilon + \epsilon');\end{aligned}$$

where the last equality follows from Lemma 12.3. Since $\alpha + \alpha' + \frac{1}{2}(\epsilon + \epsilon')$ belongs to $H(x,z)$ by Lemma 11.3, we have that $\beta + \beta' \in I^\circ$.

Now suppose $\beta'' \in I^\circ$, and let $x, y \in \mathcal{A}$, $\epsilon'' \in \Re^n$, $\epsilon \gg \mathbf{0}$, and $\alpha'' - \epsilon'' \in H(x,y)$ be such that $\beta'' = \alpha'' - \epsilon'' - \underline{\boldsymbol{q}}(x,y)$. We then have:

$$\frac{1}{2}\beta'' = \left(\frac{1}{2}\alpha'' + \frac{1}{2}\underline{\boldsymbol{q}}(x,y) - \epsilon''\right) - \underline{\boldsymbol{q}}(x,y) + \frac{1}{2}\epsilon''.$$

It follows that $\frac{1}{2}\beta'' \in I^\circ$ provided $\frac{1}{2}\alpha'' + \frac{1}{2}\underline{\boldsymbol{q}}(x,y) - \epsilon'' \in H(x,y)$. If this is the case, then we are done. Suppose otherwise; using Lemma 11.2, we conclude that $-\frac{1}{2}\alpha'' - \frac{1}{2}\underline{\boldsymbol{q}}(x,y) + \epsilon''$ belongs to the interior of the set $H(y,x)$. We now choose α and ϵ so that $-\frac{1}{2}\alpha'' - \frac{1}{2}\underline{\boldsymbol{q}}(x,y) + \frac{1}{2}\epsilon''$ is also in the interior of $H(y,x)$, and therefore $-\frac{1}{2}\alpha'' - \frac{1}{2}\underline{\boldsymbol{q}}(x,y) + \frac{1}{2}\epsilon'' - \underline{\boldsymbol{q}}(y,x) \in I^\circ$. Using Lemma 12.2, we see that

$$\begin{aligned} -\frac{1}{2}\alpha'' - \frac{1}{2}\underline{\boldsymbol{q}}(x,y) + \frac{1}{2}\epsilon'' - \underline{\boldsymbol{q}}(y,x) &= -\frac{1}{2}\alpha'' + \frac{1}{2}\epsilon'' + \frac{1}{2}\underline{\boldsymbol{q}}(x,y) \\ &= -\frac{1}{2}\beta''. \end{aligned}$$

Thus, either $\frac{1}{2}\beta'' \in I^\circ$ or $-\frac{1}{2}\beta'' \in I^\circ$. If the last case prevails, we notice $\frac{1}{2}\beta'' = \beta'' + (-\frac{1}{2}\beta'')$, and the first part of our argument shows that $\frac{1}{2}\beta'' \in I^\circ$. It follows that the set I° is mid-convex, as desired. ■

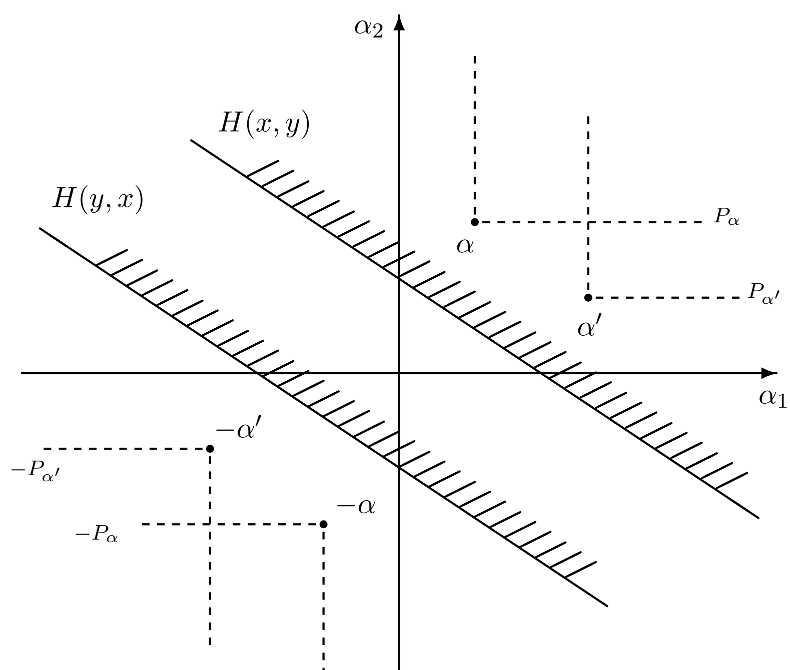

Figure 1

We are ready to prove that PAD is a sufficient condition for f to be the solution to a particular affine maximization problem. Observe that the construction of this affine maximization problem provides us with the pair (k, Q); an immediate normalization of the weights $k = (k_1, \dots, k_n)$ yields to the interpretation of f as a random dictatorship.

Proposition 14. *If the SCR $f : \mathcal{V} \to \mathcal{A}$ satisfies the positive association of differences condition, then it is a random dictatorship with respect to some pair (k, Q). That is, there exist a vector of weights $k = (k_1, \ldots, k_n)$, with $0 \le k_i \le 1$ for each $i \in \mathcal{N}$ and $\sum_i^n k_i = 1$, and a function $Q : \mathcal{A} \to \Re$ such that for every valuation profile $v \in \mathcal{V}$:*

$$\sum_{i=1}^{n} k_i\, v_i(f(v)) \ + \ Q(f(v)) \ \ \ge \ \ \sum_{i=1}^{n} k_i\, v_i(a) \ + \ Q(a), \qquad \forall\, a \in \mathcal{A}. \tag{6}$$

Proof From Lemma 13, we shall conclude that the vector $\mathbf{0}$ is a boundary point of the convex set I°. We use the Separating Hyperplane Theorem to infer that there exists a supporting hyperplane for the closure $cl(I^{\circ})$ at $\mathbf{0}$. Moreover, since $\epsilon \gg \mathbf{0}$ belongs to I°, it follows that we can choose a vector $k = (k_1, \ldots, k_n) \in \Re^n_+$, $k \ne \mathbf{0}$, such that if $\alpha \in cl(I^{\circ})$, then one has $k \cdot \alpha \ge 0$. We normalize k so that $0 \le k_i \le 1$ for each $i = 1, \ldots, n$ and $\sum_i^n k_i = 1$. Now fix an arbitrary $x_0 \in \mathcal{A}$ and define the function $Q : \mathcal{A} \to \Re$ by

$$Q(a) \ \ = \ \ \underline{q}(x_0, a), \qquad\qquad \forall\, a \in \mathcal{A}$$

where we set $\underline{q}(x_0, x_0) = 0$. We claim that the SCR f is a random dictatorship with respect to the pair (k, Q).

Indeed, consider an arbitrary type profile $v \in \mathcal{V}$ and let $y \ne f(v)$ be any alternative in $\mathcal{A}$ not chosen by f at v. Write $\alpha = v(f(v)) - v(y) \in \Re^n$. It follows that $\alpha \in H(f(v), y)$, thus $\alpha - \underline{\boldsymbol{q}}(f(v), y) \in cl(I^{\circ})$. Therefore, we can conclude:

$$\begin{aligned} 0 \ \ &\le \ \ k \cdot \big\{\alpha \ - \ \underline{\boldsymbol{q}}(f(v), y)\big\} \\ &= \ \ k \cdot v(f(v)) \ - \ k \cdot v(y) \ - \ k \cdot \underline{\boldsymbol{q}}(f(v), y). \end{aligned}$$

Using Lemma 12, we have $\underline{\boldsymbol{q}}(f(v), y) \ = \ -\,\underline{\boldsymbol{q}}(x_0, f(v)) + \underline{\boldsymbol{q}}(x_0, y)$. Replacing this last expression in the preceding equation, and re-ordering, we obtain

$$k \cdot v(f(v)) \ + \ k \cdot \underline{\boldsymbol{q}}(x_0, f(v)) \ \ \ge \ \ k \cdot v(y) \ + \ k \cdot \underline{\boldsymbol{q}}(x_0, y),$$

which is equivalent to (6) once we use the definition of the external contribution function Q and note that $k \cdot \underline{\boldsymbol{q}}(x_0, a) = \underline{q}(x_0, a) \sum_i k_i$, for all $a \in \mathcal{A}$. Since y was arbitrarily chosen, this shows that f is a random dictatorship with respect to (k, Q). ■

It is now immediate to realize that if the SCR f is randomly dictatorial, then it is dominant strategy implementable by means of a generalized Groves transfers scheme. Thus, it follows that in continuous domains, the PAD condition on the choice rules is sufficient for dominant strategy implementation.[3]

[3] This implies that in continuous domains PAD and W-MON are in fact equivalent, in the sense that a choice rule satisfies W-MON if and only if it satisfies PAD. Indeed: f is dominant strategy implementable $\Rightarrow$ f satisfies W-MON $\Rightarrow$ f satisfies PAD $\Rightarrow$ f is dominant strategy implementable.

Proposition 15. *Let $f : \mathcal{V} \to \mathcal{A}$ be a random dictatorship with respect to the pair (k, Q), and let $t : \mathcal{V} \to \Re^n$ be a generalized Groves transfer scheme. Then the direct mechanism $\Gamma = (f, t)$ is dominant strategy.*

Proof Fix an arbitrary type profile $v_{-i} \in \mathcal{V}_{-i}$ and suppose that $k_i > 0$. We show that for any valuations v_i, v_i' in $\mathcal{V}_i$, it is the case that:

$$v_i(f(v_i, v_{-i})) \;+\; t_i(v_i, v_{-i}) \;\;\geq\;\; v_i(f(v_i', v_{-i})) \;+\; t_i(v_i', v_{-i}).$$

Indeed, since t_i is a generalized Groves transfer, there exists a function $h_i : \mathcal{V}_{-i} \to \Re$ such that $t_i(\hat{v}) = \frac{1}{k_i}\big\{ \sum_{j \neq i} k_j\, \hat{v}_j(f(\hat{v})) + Q(f(\hat{v}))\big\} - h_i(\hat{v}_{-i})$, for each declared profile $\hat{v} \in \mathcal{V}$. Hence, for the given profile $v_{-i} \in \mathcal{V}_{-i}$, we have:

$$\begin{aligned} v_i(f(v_i, v_{-i})) + t_i(v_i, v_{-i}) &= \frac{1}{k_i}\Big\{ \sum\nolimits_{j=1}^{n} k_j\, v_j(f(v)) + Q(f(v))\Big\} - h_i(v_{-i}) \\ &\geq \frac{1}{k_i}\Big\{ \sum\nolimits_{j=1}^{n} k_j\, v(f(v_i', v_{-i})) + Q(f(v_i', v_{-i}))\Big\} - h_i(v_{-i}) \\ &= v_i(f(v_i', v_{-i})) + t_i(v_i', v_{-i}). \end{aligned}$$

Since this inequality holds for each $v_i, v_i' \in \mathcal{V}_i$, each $v_{-i} \in \mathcal{V}_{-i}$, and every agent $i = 1, \ldots, n$, it follows that the direct mechanism $\Gamma = (f, t)$ is dominant strategy. ■

Taken together, Propositions 9, 14, and 15 show that with continuous domains and arbitrary metric spaces as choice sets, a SCR is dominant strategy implementable iff it satisfies PAD iff it is a random dictatorship. Moreover, Proposition 15 tells us that we can use generalized Groves transfers to implement any implementable SCR. To round up a complete characterization of dominant strategy mechanisms in terms of generalized Groves mechanisms under the rich domain assumption, it remains to show that any transfer scheme implementing a SCR must be generalized Groves. This indeed can be infer by adapting a well-known result first proved by Green and Laffont [6]. We omit the details.

Proposition 16. *Let f be a dominant strategy implementable SCR. If $t : \mathcal{V} \to \Re^n$ implements f, then t must be a generalized Groves transfer scheme.*

The proof of Theorem 6 follows readily from Propositions 9, 14, 15, and 16. Note, as we mentioned earlier, that as a corollary of our characterization result we have that any dominant strategy implementable choice rule satisfies the Revenue Equivalence principle. More specifically, if $\mathcal{E}$ is a social choice environment satisfying the rich domain assumption, then given any two dominant strategy mechanisms with a common SCR, transfers implementing this choice rule are equal up to an affine transformation, since they must be generalized Groves transfers.

4 Budget balancedness

It is well-know in the literature that the implementation of an efficient SCR via Groves transfers may impose a high monetary burden to some agents. It is also known that budget balancedness is not generally satisfy. Here, we shall explore the consequences of budgetary restrictions for dominant strategy implementation in continuous domains that follow from a seemingly innocuous assumption on transfers; namely, that a uniform shift on the profile of valuations, which does not affect the equilibrium allocation, should be bear by the agents in the economy.

We start by considering a dominant strategy SCR f defined for an environment $\mathcal{E} = \{\mathcal{N}, \mathcal{A}, \mathcal{V}\}$ that satisfies the continuous domain assumption of Definition 4. From the previous analysis, it follows that f must be a random dictatorship with respect to some (k, Q). As such, it is the solution to a maximization problem where the social objective function is given by $\sum_i^n k_i \, v_i(x) + Q(x)$. We interpret the auxiliary function Q as an external contribution to society (which is in monetary terms), and consider a budgetary restriction in terms of the *net transfers* to society. Thus, if τ_i denotes the amount of transfers received by agent $i \in \mathcal{N}$, then total net transfers take the form $\sum_{i=1}^n \tau_i - Q(x)$. Weak budget balancedness in this case can be thought of as net total monetary transfers being non-positive (an alternative interpretation is to consider $-Q(x)$ be the cost incurred in the provision of the social choice $x \in \mathcal{A}$). More generally, we can write the budgetary restriction as $\sum_{i=1}^n \tau_i - Q(x) \leq b$, where b is a real number.

Definition 17. *Given a social choice environment $\mathcal{E} = \{\mathcal{N}, \mathcal{A}, \mathcal{V}\}$ consistent with the continuous domain assumption, a randomly dictatorial SCR $f : \mathcal{V} \to \mathcal{A}$ is said to satisfy the budgetary restriction with budget $b \in \Re$ if there exists a transfer scheme $t : \mathcal{V} \to \Re^n$ that implements f and for which the following holds:*

$$\sum_{i=1}^n t_i(v) \; - \; Q(f(v)) \;\; \leq \;\; b, \qquad\qquad \forall \, v \in \mathcal{V}.$$

For the sequel, we restrict the family of Groves transfer schemes that implement f to those satisfying the following condition. If the profile $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \in \mathcal{V}$ is such that for every agent $i = 1, \ldots, n$, and every alternative $a \in \mathcal{A}$, $\hat{v}_i(a) = v_i(a) + s$, where $v_i \in \mathcal{V}_i$ and s is any real number, then it must be that $h_i(\hat{v}_{-i}) = h_i(v_{-i})$. Notice that if the generalized Groves transfer scheme $t = (t_1, \ldots, t_n)$ satisfies this condition, since $f(\hat{v}) = f(v)$, it shall follow that for any $i \in \mathcal{N}$ for which $k_i > 0$, one has $t_i(\hat{v}) = t_i(v_i) + \frac{1-k_i}{k_i} s$. In other words, if each valuation is shift by a constant, then this change, positive or negative, shall be effectively pass to the society.

This condition, which may seem sensible at first, has important consequences. As our next result shows, with the provision that the external contribution function $Q \in \mathcal{C}[\mathcal{A}]$, the random dictatorship f with respect to (k, Q) will not satisfy any budgetary restriction, regardless of what limit we impose on the net deficit (or net surplus), whenever we restrict the family of generalized Groves transfers to those satisfying the above condition.

In other words, a SCR with two or more random dictators will eventually violate any budget balancedness condition that one imposes.

Proposition 18. *Let $f : \mathcal{V} \to \mathcal{A}$ be any SCR that is a random dictatorship (with at least two dictators) for the pair (k, Q), where $Q \in \mathcal{C}[\mathcal{A}]$. Then f violates the budgetary restriction with budget b, for any $b \in \Re$.*

Proof Let f be a random dictatorship for the pair (k, Q), where the vector of weights $k = (k_1, \ldots, k_n)$ satisfies $0 \le k_i \le 1$ for all $i = 1, \ldots, n$, and $\sum_i^n k_i = 1$, and further the contribution function Q is continuous on $\mathcal{A}$. Let $\mathcal{N}^*$ denote the subset of $\mathcal{N}$ for which $k_i > 0$, and assume that $\#\mathcal{N}^* \ge 2$. Fix an arbitrary budget $b \in \Re$, and consider the budgetary restriction $\sum_{i=1}^n t_i(v) - Q(f(v)) \le b$. Now, let $b' > b$; it suffices to find a profile of types $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \in \mathcal{V}$ for which the following is holds:

$$\sum_{i=1}^n t_i(\hat{v}) - Q(f(\hat{v})) = b'.$$

Suppose that $t : \mathcal{V} \to \mathcal{A}$ implements f in dominant strategies. By Proposition 16, this transfer scheme must be generalized Groves; we assume that t satisfies the aforementioned condition. Thus for each $i \in \mathcal{N}^*$, and for every valuation profile $v \in \mathcal{V}$, we write $t_i(v) = \frac{1}{k_i}\{\sum_{j \ne i} k_j v_j(f(v)) + Q(f(v))\} - h_i(v_{-i})$, where h_i is a function defined on $\mathcal{V}_{-i}$ such that if $\hat{v}_j = v_j + s$, all $j \in \mathcal{N}$, then $h_i(\hat{v}_{-i}) = h_i(v_{-i})$.

We proceed to define the following constants: $\kappa_1 = \sum_{i \in \mathcal{N}^*} \frac{1}{k_i}$, $\kappa_2 = \sum_{i \in \mathcal{N}^*} \frac{1-k_i}{k_i}$, and we observe that $\kappa_2 > 0$, since $\#\mathcal{N}^* \ge 2$. Define the function $\phi : \mathcal{A} \to \Re$ by

$$\phi(a) = \frac{b'}{\kappa_2} + \frac{1 - \kappa_1}{\kappa_2} Q(a). \tag{7}$$

We consider the profile of types $v = (v_1, \ldots, v_n)$ defined as follows: for each $i = 1, \ldots, n$, and each $a \in \mathcal{A}$, $v_i(a) = \phi(a)$. Obviously, $v \in \mathcal{V}$ by the continuous domain assumption. Denote $f(v) = x$; then for any $i \in \mathcal{N}^*$, the following holds:

$$\begin{aligned} t_i(v) &= \frac{1}{k_i}\Big\{ \sum_{j \ne i} k_j v_j(x)) + Q(x))\Big\} - h_i(v_{-i}) \\ &= \frac{1 - k_i}{k_i}\phi(x) + \frac{1}{k_i} Q(x) - h_i(v_{-i}). \end{aligned}$$

Adding up transfers to compute the net budget under profile v, we obtain from the above expression:

$$\begin{aligned} \sum_{i=1}^n t_i(v) - Q(x) &= \sum_{i \in \mathcal{N}^*} \Big\{\frac{1 - k_i}{k_i}\phi(x) + \frac{1}{k_i} Q(x)\Big\} - \sum_{i \in \mathcal{N}^*} h_i(v_{-i}) - Q(x) \\ &= \kappa_2 \phi(x) + \kappa_1 Q(x) - \sum_{i \in \mathcal{N}^*} h_i(v_{-i}) - Q(x). \end{aligned}$$

Now we use equation (7) at $x = f(v)$ to obtain, from the preceding equation:

$$\begin{aligned}\sum_{i=1}^{n} t_i(v) - Q(x) &= b' + (1-\kappa_1)Q(x) - (1-\kappa_1)Q(x) - \sum_{i\in\mathcal{N}^*} h_i(v_{-i}) \\ &= b' - \sum_{i\in\mathcal{N}^*} h_i(v_{-i}).\end{aligned}$$

Consider next the valuation profile $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)$ defined as follows: for each $i \in \mathcal{N}$ and each $a \in \mathcal{A}$, $\hat{v}_i(a) = v_i(a) + \frac{\kappa_3}{\kappa_2}$, where we set $\kappa_3 = \sum_{i\in\mathcal{N}^*} h_i(v_{-i})$. Note as well that $f(\hat{v}) = f(v) = x$. Repeating the above analysis, we obtain the following:

$$\sum_{i=1}^{n} t_i(\hat{v}) - Q(x) = b' + \kappa_3 - \sum_{i\in\mathcal{N}^*} h_i(\hat{v}_{-i}) = b',$$

where the last equality follows from the value chosen for κ_3 and the fact that $h_i(\hat{v}_{-i}) = h_i(v_{-i})$. The result now follows. ■

One shall not conclude that the provision of $Q \in \mathcal{C}[\mathcal{A}]$ is restrictive. This is because the continuous domain assumption provides a minimal richness condition for our characterization result to hold. In other words, if one considers domains of individual preferences for social alternatives that are supersets of $\mathcal{C}[\mathcal{A}]$, then Theorem 6 will still be in place.[4] Finally, a corollary of Proposition 18 is that if the aforementioned condition for the transfers is in place, the only dominant implementable mechanism that satisfies any budgetary restriction in continuous domains is the dictatorship. Notice that the presence of the external contribution function Q makes the notion of a dictator in quasi-linear environments somewhat weaker than the corresponding notion in social choice theory, since this contribution function is taken into account by the choice rule.

Corollary 19. *If f is a dictatorial social choice rule, then it is compatible with the budgetary restriction with budget b, for any $b \in \mathbb{R}$.*

Proof Let f be a dictatorial SCR with respect to the pair (k, Q). Suppose that $i^* \in \mathcal{N}$ is the dictator, so that $k_{i^*} = 1$ and $k_i = 0$ for all other $i \in \mathcal{N}$. Notice that we have $t_i(v) = 0$, for all $v \in \mathcal{V}$ and all $i \in \mathcal{N}$, $i \neq i^*$, and $t_{i^*}(v) = Q(f(v_{i^*})) - h_i(v_{-i^*})$. Here we write the dictatorial social choice rule as $f(v) = f(v_{i^*}) \in \arg\max_{x\in\mathcal{A}}\{v_{i^*}(x) + Q(x)\}$.

Fix $h_{i^*}(v_{-i^*}) \geq -b$, for any profile $v \in \mathcal{V}$. The total net transfers to society is now

$$\begin{aligned}\sum_{i=1}^{n} t_i(v) - Q(f(v_{i^*})) &= Q(f(v_{i^*})) - h_i(v_{-i^*}) - Q(f(v_{i^*})) \\ &= -h_i(v_{-i^*}) \leq b.\end{aligned}$$

Thus, the dictatorial social choice rule f can always be made to satisfy the budgetary restriction with budget b, for any $b \in \Re$. ■

[4] In fact, if one assumes that $\mathcal{V}_i$ contains all continuous and upper semi-continuous functions defined on $\mathcal{A}$, proofs of some lemmas become less involved.

References

[1] Salvador Barberà and Bezalel Peleg. Strategy-proof voting schemes with continuous preferences. *Social Choice and Welfare*, 7(1):31–38, 1990.

[2] Sushil Bikhchandani, Shurojit Chatterji, Ron Lavi, Ahuva Mu'alem, Noam Nisan, and Arunava Sen. Weak monotonicity characterizes deterministic dominant-strategy implementation. *Econometrica*, 74(4):1109–1132, 2006.

[3] Juan Carlos Carbajal. On the uniqueness of Groves mechanisms and the payoff equivalence principle. Mimeo: School of Economics, The University of Queensland, 2008.

[4] Kim-Sau Chung and Wojciech Olszewski. A non-differentiable approach to revenue equivalence. *Theoretical Economics*, 2(4):469–487, 2007.

[5] Allan Gibbard. Manipulation of voting schemes: a general result. *Econometrica*, 41(4):587–601, 1973.

[6] Jerry R. Green and Jean-Jacques Laffont. Characterization of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica*, 45(2):427–438, 1977.

[7] Birgit Heydenreich, Rudolf Müller, Marc Uetz, and Rakesh Vohra. Characterization of revenue equivalence. *Econometrica*, 77(1):307–316, 2009.

[8] Ron Lavi, Ahuva Mu'alem, and Noam Nisan. Two simplified proofs for Robert's theorem. Mimeo: Faculty of Industrial Engineering and Management, The Technion, 2008.

[9] Paul R. Milgrom and Ilya Segal. Envelope theorems for arbitrary choice sets. *Econometrica*, 70(2):583–601, 2002.

[10] Kevin Roberts. The characterization of implementable choice rules. In Jean-Jacques Laffont, editor, *Aggregation and Revelation of Preferences*, chapter 18. North-Holland, Amsterdam, The Netherlands, 1979.

[11] Michael Saks and Lan Yu. Weak monotonicity suffices for truthfulness on convex domains. *Proceedings of the 7th ACM Conference on Electronic Commerce (ACM-EC)*, 2005.

[12] Mark Allen Satterthwaite. Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(1):187–217, 1975.